AN INVESTIGATION OF NONLINEAR AEROELASTICITY IN AIRCRAFT WINGS

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Abstract

The investigation involved exploring the properties of the nonlinear spring models for aircraft wings. The theory which leads to the set of differential equations governing the motion of the aircraft wing was then described. A previous student had already made a program called wforce.f. The Runge-Kutta-Fehlberg method used in this program was described and the program was validated using examples made in past references. These were found to correspond to the results for the program. Finally an investigation was made into the effect of changing the cubic spring on the amplitude of motion of the wing after a perturbation. A harder cubic spring was found to reduce the amplitude of the limit cycle in which the wing oscillated.

1. INTRODUCTION

The response of the wing to a perturbation when in flight is critical. A perturbation caused by a gust, birdstrike or other loading could result in undesirable divergent oscillations, potentially leading to stall, higher stresses in the wing, a loss in performance of the aircraft or a combination of the three. The wing’s aeroelastic behaviour, or the “interaction of inertia, structural and aerodynamic force”, therefore ranks highly in importance when designing and analysing its performance.

The velocity at which the aircraft flies is restricted by its flutter velocity. At this velocity any perturbation will result in an oscillation with the same amplitude and no convergence. Any velocity above the flutter velocity will result in divergent oscillations, and any velocity below will result in convergence. Hence the aircraft must stay below the flutter velocity.

Normal elastic behaviour yields an elastic force, which is a linear function of the elastic displacement. Aeroelastic behaviour in wings, however, can be strongly nonlinear, with the gradient of a force-displacement curve dependent on the displacement. The nonlinearities arise both from aerodynamic and structural effects. The aerodynamic nonlinearities result from compressed flow causing shock waves with destabilise the pitching motion of the aerofoil. The structural nonlinearities arise from the nonlinear behaviour of hinges, linkages and systems in the wing as well as from the behaviour of thin wings in torsion and from wings undergoing buckling. In this project the structural nonlinearities were investigated.

The elastic linear and nonlinear properties of the wing are distributed – the displacement at a given point corresponding to a unit displacement at a certain degree of freedom is dependent on the position of the point. The distributed model can, however, be translated to a concentrated point model via mode shape functions, which will be described in more detail later in the report.

If the shape functions are known then the system can be idealised as a mass-spring-damper system and described in terms of \( n \) equations of motion, corresponding to \( n \) degrees of freedom.
These can then be rewritten to give a vector $X$ and the derivative of $X$ given by $f(X)$, and the system integrated for a desired time using various integration methods such as Houbolt’s finite difference method, or the Runge-Kutta-Fehlberg method, as used in this project.

The purpose of this project was to continue on the work of an earlier student, E. Baldassin, and confirm the functioning of his program $wforce$, written in the FORTRAN-77 language (reference 7), and investigate the behaviour of the equations for different constants of a cubic elastic term in the equations of motion.

2. THE CUBIC SPRING

The model of a wing as a purely-linear spring in rotation becomes inaccurate when nonlinearities arise due to leakages and fatigue of joints and hinges, the behaviour of thin wings in torsion, buckling of the wing, attachments to the wing such as the engine or missile pylons under fatigue and other factors. There are other models for nonlinear elastic springs. The following diagram shows a hysteresis nonlinearity, with linear behaviour exhibited up until a certain point, when the linear elastic constant suddenly changes. When the displacement is changing in the opposite direction, the change may occur at another point, such as in the diagram below:

![Figure 1. Example of hysteresis nonlinearity](image1)

Ductile materials present in systems in the wing, such as hinge mechanisms, may lead to hardening hysteresis behaviour under cyclic loading. Below is a graph of store yaw amplitude against force moment for the F-111 fighter.

![Figure 2. Store yaw amplitude vs. force moment for the F-111 aircraft](image2)
The bilinear spring is another model which consists of three regions along the displacement axis: At first, from the negative displacement, the spring has a certain elastic constant. In the second region, the system adopts another elastic constant, before returning to the first, as shown in the diagram below:

Fig. 3. General bilinear spring

In this project the investigation was concentrated on cubic spring behaviour with no hysteresis. The elastic term in the equations of motion includes a function of the displacement vector, in this case the angle of attack, raised to a power of 3. In this way the “spring” can be modelled either to soften or harden with increasing angle of attack. The equation for the elastic force is then as shown in equation (1) below:

\[ F = \beta_0 + \beta_1 \alpha + \beta_2 \alpha^2 + \beta_3 \alpha^3 \]  

(1)

Should the wing buckle, the wing will adopt the elastic characteristics of a softening spring with stiffness decreasing as the angle of rotation increases, as shown in the diagram below:

Fig. 4. Cubic softening spring
Hardening of the wing at higher angles of attack can have beneficial effects on the wing response to perturbations, enabling the wing to return to a lower angle of attack and the oscillations to occur between limit cycles instead of diverging at velocities above the flutter velocity for the wing with just a linear term. This can be a useful property for an aircraft travelling at high speed.

A thin wing undergoing torsion will most likely adopt the behaviour of a cubic spring, adopting greater stiffness as the torsion angle increases. This is shown in the diagram (figure 5) below:

**Fig. 5. Cubic hardening spring**

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![Cubic hardening spring](image)

**Fig. 6. Cubic spray with various cube parameters**

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![Cubic spray](image)
The differences in characteristics of elastic systems with differing cubic spring coefficients is shown in the figure 6 on previous page.

The graph shows the effect of hardening springs on the restoring moment, which is shown on the vertical axis. A hardening spring will produce a higher restoring moment than a simple linear spring, while a softening spring will reduce the restoring moment.

3. THEORY

In this section general analytic theory used for the project is discussed. Extensive use was made of reference [1] in this section.

3.1 Equations of motion

In this section the mathematical derivation leading up to the formulation of the differential equations for use in the program wforce.f is discussed. The program is listed in reference 7.

3.1.2 Use of shape functions

The diagram on the top of the next page shows the wing with its co-ordinate system:

![Wing coordinate system](image)

*Fig. 7. Wing coordinate system*

The following analysis uses theory taken from reference [1], pages 221 to 226:

The displacement $z$ at a position $(x,y)$ for a mode $i$ point displacement $q$ can be described through the following equation:

$$z_i(x,y,t) = \phi_i(x,y)q_i(t) \quad \text{(2)}$$

where $\phi_i(x,y)$ is the shape function for the mode $i$.

The total displacement is then given by the sum of the displacements for the individual modes, given by:

$$z(x,y,t) = \sum_{i=1}^{n} \phi_i(x,y)q_i(t) \quad \text{(3)}$$
The equations of motion can be expressed in terms of the generalised displacement vector $\mathbf{q}$ as

$$
[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F\}
$$

(4)

where $M$ is the mass matrix, $C$ the damping matrix, $K$ the stiffness matrix and $F$ the load vector.

For systems with nonlinear elastic terms the matrix $[K]$ is replaced by the matrix $[Q]$. The individual matrices $M$, $C$, $K$ and/or $Q$ for a 2 degree-of-freedom system are described as follows...

$$
[M] = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},
[C] = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix},
[K] = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}
$$

...with $Q$ taking the same form as $K$. Strictly speaking there should be a coupling damping term $C_{12}$ and $C_{21}$ but these are stated in reference [1] page 222 to be insignificant.

A mass term $M_j$ will hold the same kinetic energy for the point model moving with velocity $\dot{q}_j$ as the sum of all infinitesimal parts of with mass per unit area $m$ and area $dA$ moving with velocity $\phi_j\dot{q}_j$ and belonging to the region A. Hence

$$
M_j = \int_A m\phi_j^2 dA
$$

(5)

Similarly the damping term $C_i$ signifies a generalised damper which, at velocity $\dot{q}_i$, dissipates energy at the same rate as the sum of infinitesimal dampers which exert the local viscous pressure per unit velocity $p_{di}$ on an area $dA$ running counterphase to the local displacement velocity $\phi_i\dot{q}_i$. The local damping force is then $p_{di}\phi_i\dot{q}_i dA$ and the energy dissipation rate, or power, of the damper is the force multiplied by the local velocity. Hence the generalised damping term is given by:

$$
C_i = \int_A p_{di}\phi_i^2 dA
$$

(6)

The generalised elastic term $K_i$ or $Q_i$ absorbs the same potential energy when displaced by an amount $q_i$ as the sum of infinitesimal components along the wing absorbing potential energy when displaced by an amount $\phi_i q_i$. With the wing idealised as a beam of flexural stiffness $EI$ and length $s$ with $x$ being a displacement in the same direction as $s$, this leads to the equation for the generalised elastic term being written as:

$$
K_i = \int_s EI \left( \frac{d^2\phi_i}{dx^2} \right) dx
$$

(7)

Finally the generalised force is the force which, when displaced by a virtual displacement $\delta q_i$, does the same amount of virtual work as the local pressure $p(x,y,t)$ does at a local displacement $\phi_i\delta q_i$. Hence the generalised force is given as:

$$
F_i = \int_A p(x,y,t)\phi_i dA
$$

(8)

In terms of the motion of the wing, the displacements to be considered are the vertical “plunge” displacement and the rotational displacement: the angle of attack of the wing with respect to the airflow. The various dimensions of the wing that will be used throughout the theoretical analysis are described as in the diagram on the next page.
Fig. 8. Wing cross-section dimensions

The wing is idealised as a flat wing. Two degrees of freedom are considered: rotational (pitch) and vertical (plunge) motion.

3.1.2 The wing

The inertial force per unit span for the wing in plunge has components of both $\ddot{h}$ and $\ddot{\alpha}$, as the pitch motion has a vertical component. The vertical acceleration component due to the pitch is then $r\ddot{\alpha}$. Taking $r$ as the distance from the elastic axis, positive towards the leading edge, the inertial force per unit span for vertical motion is given by

$$\int_b^h (\ddot{h} + r\ddot{\alpha}) dm$$

$h$ and $\dot{\alpha}$ are both constant over the wing, and so the inertial term for motion in plunge can be written as $m\ddot{h} + S\ddot{\alpha}$ where

$$m = \int_b^h dm$$ and $$S = \int_b^h r dm$$, the first static moment of area.

In rotation, the inertial moment is measured in the second degree-of-freedom equation. For each infinitesimal component this is given by the inertial force multiplied by the distance from the elastic centre. Hence the inertial moment in rotation is given by

$$\int_b^h \left( \ddot{h} + r\ddot{\alpha} \right) dm = I_\alpha \ddot{\alpha} + S\dot{h}$$

Where $I_\alpha$ is the moment of inertia of the wing about the elastic axis.

In damping, for the plunge motion the damping coefficient can be written as $C_h$ and for the rotational motion it can be written as $C_\alpha$.

$\overline{G}(h)$ is the elastic term in plunge and $\overline{M}(\alpha)$ the elastic term is pitch. In this project $\overline{M}(\alpha)$ takes on a nonlinear form, given by the equation

$$\overline{M}(\alpha) = \beta_0 + \beta_1 \alpha + \beta_2 \alpha^2 + \beta_3 \alpha^3$$  \[9\]
Finally, \( p(t) \) and \( r(t) \) are the applied vertical force and rotational moment respectively. The equations of motion can now be written as:

\[
\begin{align*}
\dot{m} \ddot{h} + S \ddot{\alpha} + C_e \dot{h} + \ddot{G}(h) &= p(t) \tag{10} \\
\ddot{S} \dot{\alpha} + I_\alpha \ddot{\alpha} + C_e \alpha + \ddot{M}(\alpha) &= r(t) \tag{11}
\end{align*}
\]

\( p(t) \) and \( r(t) \) are composed of the lift and pitching moment as well as any extra external forces applied through hinges and any other sources.

The system of equations was then written in nondimensional form. In order to achieve this several new variables were introduced as follows:

\( \omega_\xi \) and \( \omega_\alpha \), the natural frequencies of the plunge and pitch “spring systems” respectively were defined as:

\[
\omega_\xi = \sqrt{\frac{K_\xi}{m}} \quad \text{and} \quad \omega_\alpha = \sqrt{\frac{K_\alpha}{I_\alpha}}
\]

with \( K_\xi \) and \( K_\alpha \) being the plunge and pitch linear spring constants respectively.

\[
r_\alpha = \sqrt{\frac{I_\alpha}{mb^2}}
\]

\( r_\alpha \) giving the radius of gyration about the elastic axis

\[
\mu = \frac{m}{\pi \rho b}
\]

\( \mu \) giving the air-wing density ratio

\[
x_\alpha = \frac{S}{mb}
\]

With the substitutions made, the equations can then be simplified as in ref [1] to

\[
\xi^* + x_\alpha \alpha^* + 2 \zeta \xi + \frac{\sigma}{U^*} \xi^* + \left( \frac{\sigma}{U^*} \right)^2 G(\xi) = -\frac{1}{\pi \mu} C_L (\tau) + \frac{P(\tau) b}{m U^2}, \tag{12}
\]

\[
\frac{x_\alpha}{r_\alpha^2} \alpha^* + \frac{\alpha^*}{U^*} + \frac{1}{U^*} M(\alpha) = \frac{2}{\pi \mu} C_M (\tau) + \frac{Q(\tau)}{m U^2 r_\alpha^2}, \tag{13}
\]

where \( G(\xi) = \ddot{G}(h) / K_\xi \) and \( M(\alpha) = \ddot{M}(\alpha) / K_\alpha \) and where \( \dot{\alpha}, \ddot{\xi}, \ddot{\alpha}, \ddot{\xi} \), the derivatives in normal time, are replaced by \( \dot{\alpha}' , \ddot{\xi}' , \ddot{\alpha}' , \ddot{\xi}' \), the derivatives in nondimensional time.

The velocity, angular velocity and time were also converted to give non-dimensional variables, given by:

\[
U^* = \frac{U}{b \omega_\alpha}, \quad \sigma = \frac{\omega_\xi}{\omega_\alpha} \quad \text{and} \quad \tau = \frac{U t}{b}
\]

\( P \) and \( Q \) are respectively the externally-applied forces and moments, while the lift and moment coefficients \( C_L \) and \( C_M \) were given by Fung [5] as:

\[
C_L (\tau) = \pi \left( \xi^* - a_0 \alpha^* + \alpha' \right) + 2 \pi \left[ \alpha(0) + \xi'(0) + \left[ \frac{1}{2} - a_h \right] \alpha'(0) \right] \varphi(\tau)
\]

\[
+ 2 \pi \int_0^\sigma \varphi(\tau - \sigma) \left[ a'_*(\sigma) + \xi'^*(\sigma) + \left[ \frac{1}{2} - a_h \right] a^*(\sigma) \right] \sigma
\]

\( \varphi(\tau) \) is a function of \( \tau \) and

\( \alpha(\sigma) \) and \( \xi'(\sigma) \) are functions of \( \sigma \).
where \( \varphi(\tau) \) is the Wagner function given by:

\[
\varphi(\tau) = 1 - \psi_1 e^{-\kappa_1 \tau} - \psi_2 e^{-\kappa_2 \tau}
\]

where \( \psi_1 = 0.165 \), \( \psi_2 = 0.335 \), \( \kappa_1 = 0.0455 \) and \( \kappa_2 = 0.3 \).

The integral terms in the above equations for \( C_m \) and \( C_L \) are difficult to integrate numerically. Lee [6] introduced four new variables \( w_1 \) to \( w_4 \), defined below:

\[
\begin{align*}
w_1 &= \int e^{-\xi(\tau-\sigma)} \alpha(\sigma) d\sigma \\
w_2 &= \int e^{-\xi(\tau-\sigma)} \alpha'(\sigma) d\sigma \\
w_3 &= \int e^{-\xi(\tau-\sigma)} \xi(\sigma) d\sigma \\
w_4 &= \int e^{-\xi(\tau-\sigma)} \xi'(\sigma) d\sigma
\end{align*}
\]

Equations (12) and (13) can now be written in general form as (from reference [1]):

\[
\begin{align*}
&c_0 \xi^2 + c_1 \alpha^2 + c_2 \xi + c_3 \alpha + c_4 \xi + c_5 \alpha + c_6 w_1 + c_7 w_2 + c_8 w_3 + c_9 w_4 \\
&+ \left( \frac{\sigma}{U^*} \right)^2 G(\xi) = f(\tau)
\end{align*}
\]

\[
\begin{align*}
&d_0 \xi^2 + d_1 \alpha^2 + d_2 \alpha + d_3 \xi + d_4 \xi + d_5 w_1 + d_6 w_2 + d_7 w_3 + d_8 w_4 \\
&+ \left( \frac{1}{U^*} \right)^2 M(\alpha) = g(\tau)
\end{align*}
\]

The constants \( c_1 \) to \( c_9 \), along with \( d_1 \) through to \( d_9 \) are shown in appendix A. If \( P = Q = 0 \), the equations for \( f(\tau) \) and \( g(\tau) \) are as below:

\[
\begin{align*}
f(\tau) &= \frac{2}{\mu} \left( \frac{1}{2} - a_h \right) \alpha(0) + \xi(0) \left( \psi_1 e^{-\kappa_1 \tau} + \psi_2 e^{-\kappa_2 \tau} \right) \\
g(\tau) &= -\frac{(1+2a_h)}{2\mu^2} \frac{f(\tau)}{a_h^2}
\end{align*}
\]

A vector \( X = \{x_1, x_2...x_8\}^T \) is then introduced, such that

\[
x_1 = \alpha, \quad x_2 = \alpha', \quad x_3 = \xi, \quad x_4 = \xi', \quad x_5 = w_1, \quad x_6 = w_2, \quad x_7 = w_3, \quad x_8 = w_4,
\]
Equations (18) and (19) can then be expressed as a differential of the vector $X$:

$$\dot{X} = f(X)$$  \hspace{1cm} (22)

The differential of the term $w_1$ above is given as $w_1' = \alpha - \varepsilon_1 w_1$, and the derivatives of $w_2$, $w_3$ and $w_4$ can be similarly derived.

Treating equations (18) and (19) as a pair of simultaneous equations, the individual differential terms then are given by

$$x_1' = x_2, \quad x_2' = (c_0 H - d_0 P)/(d_0 c_1 - c_0 d_1), \quad x_3' = x_4, \quad x_4' = (-c_1 H + d_1 P)/(d_6 c_1 - c_0 d_1),$$

$$x_5' = x_6', \quad x_6' = x_7 - \varepsilon_2 x_7, \quad x_7' = x_8 - \varepsilon_3 x_8, \quad x_8' = x_9 - \varepsilon_2 x_9$$  \hspace{1cm} (23)

where

$$P = c_2 x_1 + c_3 x_2 + c_4 x_3 + c_5 x_4 + c_6 x_5 + c_7 x_6 + c_8 x_7 + c_9 x_8 + \left(\frac{\varepsilon}{U^*}\right)^2 G(x_5) - f(x_5)$$  \hspace{1cm} (24)

$$H = d_2 x_3 + d_3 x_4 + d_4 x_5 + d_5 x_6 + d_6 x_7 + d_7 x_8 + d_8 x_9 + d_9 x_9 + \left(\frac{\varepsilon}{U^*}\right)^2 M(x_5) - g(x_5)$$  \hspace{1cm} (25)

3.2 Bifurcation analysis

The definition of a bifurcation is a change in the characteristics of a graph of a parameter of a nonlinear system against one of its vectors. Bifurcations come in many forms. The Hopf bifurcation occurs when the vector state changes from convergence to a single point to a limit cycle oscillation. A period-doubling bifurcation occurs when a vector oscillating in a certain number of limit cycles changes to existing in twice that number of limit cycles. The period-tripling bifurcation is analogous to the period-doubling bifurcation but with the number of limit cycles tripling. This is also the case for greater number period-multiplying bifurcations.

The hopf bifurcation points can be found for a linear system of differential equations where $\dot{X} = f(X)$ as follows:

Let $X_E$ be a fixed point in the system. When the vector of $X$ is $X_E$ then $\dot{X} = 0$ and the only way for the vector to change is if a perturbation is applied. Let this perturbation be written as $y(\tau).$ Then $y(\tau)$ can be expressed as:

$$y(\tau) = X(\tau) - X_E$$  \hspace{1cm} (27)

The rate of growth/decay of this perturbation can then be given by the following equation:

$$\dot{y} = \frac{d}{d\tau}(X - X_E) = \dot{X} - f(X) = f(X_E + y)$$  \hspace{1cm} (28)

The term $f(X_E + y)$ can be expanded about $X_E$ using the Taylor series to the first order:

$$f(X_E + y) = f(X_E) + \frac{\partial f}{\partial X}|_{X_E} y + O(y^2)$$  \hspace{1cm} (29)

Since $f(X_E) = 0$ since at this point $|\dot{X} = 0|$, $\dot{y}$ is given by

$$\dot{y} = \frac{\partial f}{\partial X}|_{X_E} y = Jy$$  \hspace{1cm} (30)

where $J$ is the Jacobian matrix of $f(X)$.

If every eigenvalue of $J$ is non-zero and has a negative real part, then the system is stable. If, however, one eigenvalue has a positive real part, then the system will grow exponentially.

**AN INVESTIGATION OF NONLINEAR AEROELASTICITY IN AIRCRAFT WINGS**
3.3 Integrating schemes

3.3.1 Runge-Kutta method

The differential equation for the state space system is written below:

\[ \frac{d\mathbf{X}}{dt} = f(\mathbf{X}, t) \]

In order to evaluate this computationally a Fehlberg Method of order 4 was used. More detail on this can be found in reference [4].

3.3.2 Fehlberg Method of Order 4

This method is a variation on the Runge-Kutta method of order 4. With the initial conditions \( \mathbf{X}_0 \) known at time \( t=0 \), the Fehlberg method is used to find the value of \( \mathbf{X} \) at time \( t+dt \), where \( dt \) is a small increment. The function \( f \) is evaluated in this interval for five points in time, the values at those points being denoted by \( f_{k_1} \) through to \( f_{k_5} \), and five hypothetical values of \( \mathbf{X} \) based on the \( f \)-value for the previous point, as shown below:

\[
\begin{align*}
  f_{k_1} &= f(x, t) \\
  f_{k_2} &= f\left(x + \frac{\Delta t}{4} f_{k_1}, t + \frac{\Delta t}{4}\right) \\
  f_{k_3} &= f\left(x + \frac{3\Delta t}{32} f_{k_1} + \frac{9\Delta t}{32} f_{k_2}, t + \frac{3\Delta t}{8}\right) \\
  f_{k_4} &= f\left(x + \frac{1932\Delta t}{2197} f_{k_1} - \frac{7200\Delta t}{2197} f_{k_2} + \frac{7296\Delta t}{2197} f_{k_3}, t + \frac{12\Delta t}{13}\right) \\
  f_{k_5} &= f\left(x + \frac{439\Delta t}{216} f_{k_1} - \frac{2517\Delta t}{216} f_{k_2} + \frac{13514\Delta t}{216} f_{k_3} - \frac{5039\Delta t}{216} f_{k_4}, t + \frac{1\Delta t}{5}\right)
\end{align*}
\]

The value of \( \mathbf{X} \) at time \( t = t+dt \) can now be found as follows:

\[
X(t+\Delta t) = X(t) + \Delta t \left( \frac{25}{216} f_{k_1} + \frac{1408}{2565} f_{k_2} + \frac{2197}{4104} f_{k_3} - \frac{1}{5} f_{k_4} \right)
\]

The Runge-Kutta-Fehlberg method is accurate to the order of \( \Delta t^5 \). This compares well with other methods used, such as Houbolt’s finite difference scheme, as described in reference [7], which is accurate only to order \( \Delta t^4 \), although it is less efficient than Houbolt’s method, which was used by Lee et al to calculate the reference graphs in the results section. The time step used as quoted in Lee [1] was quoted as being sufficiently small when having a value of 1/256. In this investigation a time-step of 0.001 was used, making it even more accurate.

4. VERIFICATION OF PROGRAM

The program wforce was verified using a series of tests to see whether it corresponded with the results given in reference [1].

4.1 Vector-time diagrams

The first test consisted of running the program with an initial condition of \( x_1 = 1^\circ \) with \( x_2 \) through to \( x_8 \) set to zero, and with the parameters set as in the table below:

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( x_a )</th>
<th>( a_h )</th>
<th>( r_a )</th>
<th>( \varpi )</th>
<th>( \zeta_a )</th>
<th>( U^*/U_{*L} )</th>
<th>( \beta_1 ) (1/\text{rad})</th>
<th>( \beta_3 ) (1/\text{rad})</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>-0.5</td>
<td>0.5</td>
<td>0.2</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
The iterations were run for 200 seconds. The output from the program wforce was given for \( \alpha \) against \( t \) and for \( \xi \) against \( t \) and compared to the results given in reference [2], figures 10 and 11. Note that the results for reference [2] were calculated using Houbolt’s finite difference method instead of a Runge-Kutta-Fehlberg method, as in the program wforce.

Below is the graph of \( \alpha \) versus \( t \) from the program:

![Graph of \( \alpha \) versus \( t \)](image)

**Fig. 9. Output from wForce F program**

The amplitude decreases with time and converges to zero, as would be predicted for a system below the linear flutter velocity.

The comparison output from reference [2] is shown below.

![Graph of \( \xi \) versus \( t \)](image)

**Fig. 10. Output from ref 3 corresponding to wforce.F output**

The two graphs are seen to correspond very well, with the period, amplitude and phase being the same in both figures for the same moments in time.
Below is the graph of $\xi$ versus $t$ from the program:

![Graph of $\xi$ versus $t$ from the wforce.F program](image)

Fig. 11. $\xi$ versus $t$ from the wforce.F program

This compares to figure 11 in reference [2], as shown above.

![Graph from reference 2 with parameters corresponding to Fig. 11](image)

Fig. 12. Graph from reference 2 with parameters corresponding to Fig. 11

The results are seen to compare almost exactly. Initially there are large oscillations of plunge, with the amplitude initially increasing, but then levelling out due to the velocity being well below the linear flutter velocity and due to the cubic spring factor.

4.2 Bifurcation diagrams

A bifurcation diagram was produced showing the behaviour of $\alpha$ against $U^*/U_L^*$ (where $U_L^*$ is the linear flutter velocity for a system with the relevant parameters, a cubic spring constant of zero and a linear spring constant of 1). This diagram proceeded from $U^*/U_L^* = 0.1$ to $U^*/U_L^* = 1.0$, with initial conditions of $\alpha(0) = 7$. The parameters used are shown in the table on the next page.
Tab. 2.

<p>| | | | | | | |</p>
<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
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<td>$\alpha_0$</td>
<td>$\alpha_h$</td>
<td>$r_\alpha$</td>
<td>$\omega$</td>
<td>$\xi_\alpha$</td>
<td>$\xi_\eta$</td>
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<td>-0.5</td>
<td>0.5</td>
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<td>0</td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>40</td>
</tr>
</tbody>
</table>

The integration was carried out for 400 seconds at each value of $U^*/U_{*L}$ with intervals of $U^*/U_{*L} = 0.01$. The initial conditions for the next value of $U^*/U_{*L}$ were taken as the final X-vector at 400 seconds for the preceding $U^*/U_{*L}$. The diagram then is as follows on the top of the next page.

Fig. 13. $\alpha$ vs. $U^*/U_{*L}$ for parameters in table 2

It is seen that a Hopf bifurcation occurs at $U^*/U_{*L}$, followed by a single limit cycle, with a “period-tripling” cycle occurring at $U^*/U_{*L} = 0.83$. The limit cycle amplitudes then diverge towards higher values of $\alpha$. It is noted that the oscillations in this region are above 10°. This has two implications: Since the stalling angle for most aerofoils lies between 10° and 15°, it would not be desired for the aircraft to fly at any higher value of $U^*/U_{*L}$. Secondly, since in the equations of motion small angles of attack are assumed for the purposes of finding the $C_L$ with a linear lift-curve slope, these equations would become increasingly inaccurate with $\alpha$ becoming large.

The graph compares to figure 3 in reference [3], as shown below:

Fig. 14. Graph in reference 3 using parameters from Fig. 13
The solid line shows the analytical results predicted using a describing function, as described on page 180 of reference [3].

Next another bifurcation diagram was produced to compare results between the program and with reference [3] for different parameters and a harder spring, with a lower elastic term. This proceeded at values of $U^*/U_*^*$ from $U/U_*^* = 0.1$ to $U^*/U_*^* = 0.6$, with intervals of $U^*/U_*^* = 0.002$. The time step was 0.001, with the integration period being 1200 between $U^*/U_*^* = 0.14$ and $U^*/U_*^* = 0.18$, 1600 between $U^*/U_*^* = 0.27$ and $U^*/U_*^* = 0.31$, 2400 between $U^*/U_*^* = 0.44$ and $U^*/U_*^* = 0.52$, and 200 seconds elsewhere. The integration period was extended in the regions specified because it was earlier found that key changes in the limit cycle patterns took place in those regions, and hence a longer settling time might be expected. The input was $\alpha(0) = 3^\circ$.

The parameters for the calculation were as follows:

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$x_\alpha$</th>
<th>$a_h$</th>
<th>$r_\alpha$</th>
<th>$\alpha$</th>
<th>$\zeta$</th>
<th>$\zeta_\xi$</th>
<th>$\beta_1$ (1/rad)</th>
<th>$\beta_3$ (1/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.25</td>
<td>-0.5</td>
<td>0.5</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0.01</td>
<td>50</td>
</tr>
</tbody>
</table>

The bifurcation diagram as created from data calculated by the program is shown below:

![Bifurcation Diagram](image)

**Fig. 15. $\alpha$ vs. $U/U^*$ for parameters in table 2**

The comparison diagram from reference [3] for the same input parameters is shown below:

![Comparison Diagram](image)

**Fig. 16. Graph from reference 3 for parameters used in Fig. 15**
Both diagrams show a Hopf bifurcation at $U^*/U^*_L$ (denoted $U/U^*$ in the reference) = 0.13. A period-doubling bifurcation also takes place at $U^*/U^*_L = 0.27$. At $U^*/U^*_L = 0.425$ more period-doubling bifurcations take place at each limit cycle, with further period-doubling bifurcations taking place at $U^*/U^*_L = 0.438$. Where there are scattered points in the diagram, chaos is present. Localised chaos within the existing limit cycles is seen to take place at $U^*/U^*_L = 0.47$, with the chaotic region spreading to cover the majority of the pitch amplitude at $U^*/U^*_L = 0.81$. The Lyapunov exponent can be used to check for the chaotic behaviour of a system. If the Lyapunov exponent is negative or zero, the system is non-chaotic. However, if the exponent is positive, then chaotic behaviour is present. Price et al in reference [3] computed the Lyapunov exponents along the diagram. They found that for various velocities on the diagram the exponent was negative or zero. However, for $U^*/U^*_L = 0.47$, the exponent was 0.01, indicating chaos.

There are some differences between the two diagrams. These can be ascribed to the different methods of integration used and, more importantly, to different attractors being followed. After the final time iteration for each value of $U^*/U^*_L$, the values of $x_1$ through to $x_8$ are taken to be the initial conditions for the new value of $U^*/U^*_L$.

The paths between $U^*/U^*_L = 0.3$ and $U^*/U^*_L = 0.5$ follow the limit cycles only because the initial conditions for each value of $U^*/U^*_L$ dictate that these should be followed. Were one or more of the eight initial conditions $x_1$ through to $x_8$ changed to be the negative of the original one, the limit cycle might change to be centred on another attractor. The table of initial conditions at $U^*/U^*_L = 0.305$ is shown below:

<table>
<thead>
<tr>
<th></th>
<th>x(1)</th>
<th>x(2)</th>
<th>x(3)</th>
<th>x(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>-0.02416</td>
<td>0.00492</td>
<td>0.09652</td>
<td>0.01007</td>
</tr>
<tr>
<td>x2</td>
<td>-0.74867</td>
<td>-0.11479</td>
<td>-0.07438</td>
<td>0.20067</td>
</tr>
</tbody>
</table>

From $x_1$ to $x_4$, each initial condition was changed and the program run for the same time just at $U^*/U^*_L = 0.305$. The results of $x_1$ against $x_2$ (or $\alpha$ against $\dot{\alpha}$) for each case were recorded and a 2D state-space diagram produced for those conditions. First the diagram with all initial conditions as normal was produced, and this is shown below:

**Fig. 17. 2D state-space diagram with initial conditions as in table 4**

This shows limit cycles in addition to the high-amplitude limit cycles present just to the right of $\alpha=0^\circ$. Next, the value of $x_1$ was changed to the negative of its original. This resulted in the following state space diagram:
As can be seen the outer limit cycle on the right loses some stability and the points diverge. Next, the value of $x_2$ was changed to its negative, in addition to $x_1$ already being the negative of its original, shown on the next page.

The right-side limit cycle becomes slightly less stable than before, however the limit cycles are still centred around the same points. With $x_3$ changed as well, the shape of the diagram became completely different, as shown below:

The limit cycle has switched sides, with the smaller limit cycles now moving to the left of $\alpha = 0^\circ$. The different initial conditions meant that the trajectory was able to converge to a different path. Finally, with $x_1$, $x_2$, $x_3$ and $x_4$ all the negative of their original values, this resulted in the diagram on the next page.
This shows slightly less condensed paths indicating less stability and attraction to the main limit cycle paths.

With $x_1$, $x_2$ and $x_3$ changed as above, another series was produced between $U^*/U^*_{L} = 0.305$ and $U^*/U^*_{L} = 0.5$. This is shown in purple in the diagram below:

**Fig. 22. α vs. $U/U^*$ diagram with sign change of $x_1$ to $x_3$ in table 3**

The results for the modified initial conditions at $U/U^* = 0.3$ are seen to correspond to the reference bifurcation diagram in figure 16.

### 4.3 Poincare map

The Poincare map can be used to check for chaotic behaviour by looking at the distribution of points on an $\alpha - \dot{\alpha}$ diagram. Poincare maps can be produced by various methods and produce differing outputs depending on what criteria were used to produce them. In this case data was recorded at the points where $\xi$ reached a maximum or a minimum as a function of time, at $U^*/U^*_{L} = 0.47$, with the initial conditions $\alpha = 3^\circ$ and $x_2$ through to $x_8 = 0$, and the parameters as for figures 15 through to 22. On the top of the next page is the Poincare map for the parameters (figure 23).

The Poincare map shows scattered points which have are non-periodic and chaotic in the regions where the limit cycles used to exist. For periodic behaviour the points in each part of the diagram would have coincided with each other as with each period repetition the alpha and alpha dot would have had the same value.
5. OTHER RESULTS

For just a linear spring present, the system is linear and has decaying oscillations right up until \( U^*/U^*_{\text{L}} = 1.0 \), after which the oscillations become divergent.

For a linear spring, the system converges to \( \alpha=0^\circ \) from \( U^*/U^*_{\text{L}} = 0 \) to \( U^*/U^*_{\text{L}} = 1 \). At \( U^*/U^*_{\text{L}} = 1 \), the system changes to diverging towards infinity. This means that, if the integration period was long enough, the value of \( \alpha \) would go towards infinity.

Another type of bifurcation diagram was made for the final investigation. The behaviour of \( \alpha \) against \( \beta_3 \) was investigated for \( \beta_3 \) decreasing from 100, at intervals of 0.5, for the general parameter values as in the table below:

<table>
<thead>
<tr>
<th>Tab. 5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
</tr>
<tr>
<td>200</td>
</tr>
</tbody>
</table>

The bifurcation diagrams were made with \( \beta_3 \) decreasing and not increasing because it was soon discovered that as the value became smaller, the limit cycles grew larger in amplitude. It is best to start at the smallest value so that the values do not converge to zero and so that behaviour can be seen at larger amplitudes. For all cases the input conditions were \( \alpha(0) = 3^\circ \), with all the other variables set to 0.

For a linear spring constant of \( \beta_1 = 0.01 \) and at \( U^*/U^*_{\text{L}} = 0.2 \), the bifurcation diagram was made with the varying parameter being \( \beta_3 \). The diagram is shown on top of the next page (figure 24).

There are no bifurcations present in the diagram, instead the limit cycles merely diverge as the value of \( \beta_3 \) tends to 10. At around \( \beta_3 = 30 \), the gradient of the curve becomes significant. It would therefore be preferable to have a \( \beta_3 \) value larger than 30 which would make the limit cycle smaller. The initial perturbation was \( \alpha(0)=3^\circ \). It was seen that increasing \( \alpha(0) \) merely had the effect of making the limit cycles bigger in a proportionate manner, with the shape staying constant. The diagram was originally investigated for values of \( \beta_3 \) down to \( \beta_3 = 0 \). It was found, however, that at these low values of \( \beta_3 \) the oscillations increased in the same manner as before, with
the steepness of the curve rapidly increasing and the limit cycles growing unreasonably-large to be displayed on the diagram.

\[ \text{Fig. 24. variation of } \alpha \text{ vs. } \beta_3 \text{ for } \beta_1 = 0.01 \text{ and } U^*/U^*_L = 0.2 \]

Next, the system was investigated for \( \beta_1 = 1 \) and \( U^*/U^*_L = 1.0 \). This is significant as this is the flutter velocity for the linear spring constant chosen. The diagram was made for values of \( \beta_1 \) from 100 down to \(-50\), and the results are shown in the graph below:

\[ \text{Fig. 25. variation of } \alpha \text{ vs. } \beta_3 \text{ for } \beta_1 = 1 \text{ and } U^*/U^*_L = 1.0 \]

The oscillations are seen to rapidly die down from their initial perturbation to a settled value of around \( \alpha = \pm/0.31^\circ \), and then stay fixed in a straight line. Having a large cubic coefficient is hence seen to have no effect on the oscillations, hence making them independent of it. Closer analysis of the numerical results shows the reduction in oscillation amplitude at a given cubic term near 100 to be small, with the bulk of the reduction taking place due to the cubic term decreasing. This means that were the cubic term to stay constant, one should not expect such a damped oscillation, or indeed any damping, as might initially seem from the diagram above.

Finally the behaviour was investigated at \( \beta_1 = 0.01 \) and \( U^*/U^*_L = 1.0 \). The diagram is shown on the next page.
It is seen that there are three limit cycles present with amplitudes increasing to $\alpha=15^\circ$ at $\beta_3 = 30$. An aircraft in flight with these parameters would be dangerous to fly in, as any perturbation will result in the angle of attack going dangerously high, decreasing the amount of control that the pilot has on the aircraft and bringing the angle of attack too close to the stalling angle for the aerofoil.

6. CONCLUSION

Nonlinear behaviour of the type covered in this project is commonplace in aircraft wings. It can not only affect the velocity at which flutter occurs but also the motion resulting from perturbations to it. The presence of limit cycle oscillations in a nonlinear system has the potential, if not corrected, to cause fatigue of the wing through repeated loading and unloading of the wing if not corrected. Added to this a loss in aerodynamic performance is to be expected through the deviation from optimal angle of attack for flight, leading to an increase in drag, lift and hence possibly other aircraft modes of oscillation such as the phugoid motion. The correction of a perturbation in a limit cycle may lead to greater work for the pilot and, if the period of oscillation is of the same magnitude as the pilot’s reaction time, to potential reinforcement of the oscillation amplitude through the pilot applying force at the wrong time in the period. Ideally the period of oscillation should be long enough for the pilot to be able to judge when to apply the correcting force on the stick.

In a linear system it is the linear flutter velocity that is the most important value. For the nonlinear system the critical point to consider is the Hopf bifurcation point. It is at this point that the system ceases to converge to a fixed point and instead oscillates in a limit cycle between two extreme values. This Hopf bifurcation point is found for linear systems using the Jacobian matrix of the derivative equation as described in section 3.2.

It is also important to know the effects of having different cubic spring coefficients on the behaviour of the system. These were investigated in this project through making a bifurcation diagram, with the finding being that at higher values of $\beta_3$ the amplitude of the limit cycle oscillations is lower. The Hopf bifurcation points for bifurcation diagrams of $\alpha$ against $U*/U*_{L}$ would, of course, have to be found as well. If these have a lower value of $U*/U*_{L}$ for a higher value of $\beta_3$ then the overriding requirement for safety and controllability at higher values of $U*/U*_{L}$ would mean that it would be preferable for the wing to have less hardening properties.

The program was tested against references [2] and [3] and the results were found to correspond. The bifurcation diagram in this report in figure 15 was even found to be more accurate.
than the corresponding figure for the reference because the limit cycle boundaries followed the same path instead of switching paths, as they did in figure 16.

As has been discussed, nonlinearities arise not only from the hardening of the wing in torsion but also from linkages, hinges, flaps and other systems in the wing. This was not taken into account in the analysis here. The nonlinearities which these systems possess tend to be bilinear or hysteresis nonlinearities. In order for this to be taken into account extra lines would have to be written in the wforce program to incorporate extra terms into the elastic term depending on the value of $\alpha$, using IF statements to input different equations for restoring moments for different values of $\alpha$.

Another improvement on the program would be to make use of a more efficient routine than the Runge-Kutta-Fehlberg method. Houbolt’s finite difference scheme is described on page 239 of reference [1] and could be made use of.

The results produced have been for cases with zero structural damping. With damping present, the oscillations would decrease quicker and the limit cycles would be of a lower amplitude due to some of the energy being absorbed by the dampers. The dampers may also exert some influence over the Hopf bifurcation velocity in a $\alpha$ vs $U*/U*_{L}$ bifurcation diagram.

Overall the results, as well as confirming the validity of the program wforce, provide a good insight into the response of the system to perturbations, and are useful in determining the properties of the limit cycle oscillations and the properties of the aircraft in flight. They lay a good basis for expanding the study in the future.

7. REFERENCES


Glossary of symbols used

$f(X)$ - function which gives derivative in time of $X$

$Mi$ - Generalised mass term for the $i$ degree of freedom

$Ci$ - Generalised damping term for the $i$ degree of freedom

$Ki$ - Generalised elastic term for the $i$ degree of freedom

$Fi$ - Generalised force term for the $i$ degree of freedom

$\beta i$ - Coefficient of $\alpha i$ in nonlinear elastic pitch term.

$\bar{M}$ - Nonlinear elastic restoring moment term.

$\mu$ - aerofoil/mass ratio ($m/\pi\rho b^2$)

$zi$ - Displacement perpendicular to xy plane for mode $i$

$\phi i$ - Mode shape function for mode $i$
qi - Point displacement for degree of freedom i
EI - Flexural stiffness of wing
h - plunge distance
α - angle of attack
ξ - Nondimensional plunge distance (h/b)
m - mass per unit span of wing
S - First moment of area per unit span of wing about elastic axis
rα - Radius of gyration about elastic axis
ω - angular velocity (subscript dictates mode)
U* - Nondimensional velocity (U/bωα)
τ - Nondimensional time (tU/b)
ωα - Nondimensional angular velocity (ωξ/ωα)
CL - Lift coefficient
CM - Pitching moment coefficient
ah - nondimensional distance from aerofoil midchord to elastic axis
φ(τ) - Wagner function at time τ
b - aerofoil semi-chord
c - aerofoil chord

Appendix A

Given

\[ \xi'' + x_0 \alpha^2 + 2 \xi \frac{\omega}{U^*} \xi' + \left( \frac{\omega}{U^*} \right)^2 G(\xi) = \frac{1}{\pi \mu} C_L(\tau) + \frac{P(\tau)b}{mU^2}, \]

and

\[ \frac{x_0}{r^2} \xi'' + \alpha^2 + 2 \frac{\xi \omega}{U^*} + \frac{1}{U^{*2}} M(\alpha) = \frac{2}{\pi \mu^2} C_M(\tau) + \frac{Q(\tau)}{mU^2r^2} \]

with

\[ C_L(\tau) = \pi \left( \xi'' - a_h \alpha'' + \alpha' \right) + 2 \pi \left[ \alpha(0) + \xi'(0) + \left( \frac{1}{2} - a_h \right) \alpha'(0) \right] \varphi(\tau) \]

\[ + 2 \pi \int_0^\infty \varphi(\tau - \sigma) \left[ \alpha'(\sigma) + \xi''(\sigma) + \left( \frac{1}{2} - a_h \right) \alpha''(\sigma) \right] d\sigma \]

and

\[ C_M(\tau) = \pi \left( \frac{1}{2} + a_h \right) \left[ \alpha(0) + \xi'(0) + \left( \frac{1}{2} - a_h \right) \alpha'(0) \right] \varphi(\tau) \]

\[ + \pi \left( \frac{1}{2} + a_h \right) \int_0^\infty \varphi(\tau - \sigma) \left[ \alpha'(\sigma) + \xi''(\sigma) + \left( \frac{1}{2} - a_h \right) \alpha''(\sigma) \right] d\sigma \]

\[ + \pi \left( \frac{1}{2} - a_h \right) \int_0^\infty \varphi(\tau - 2\pi \tau) \left[ \alpha'(2\pi \tau) + \xi''(2\pi \tau) + \left( \frac{1}{2} - a_h \right) \alpha''(2\pi \tau) \right] d\sigma \]

\[ - \left( \frac{1}{2} - a_h \right) \frac{\pi^2}{2} \alpha - \frac{\pi^3}{16} \alpha'' \]

the equations can be simplified by substituting the equations in for \( C_L \) and \( C_M \) and rearranging. The equations then simplify to:

\[ c_0 \xi'' + c_1 \alpha^2 + c_2 \xi' + c_3 \alpha + c_4 \xi + c_5 \alpha + c_6 \xi + c_7 \alpha + c_8 \xi + c_9 \alpha + \left( \frac{\omega}{U^*} \right)^2 G(\xi) = f(\tau) \]

and
\[ d_0 \xi'' + d_1 \alpha'' + d_2 \alpha + d_3 \xi + d_4 w_1 + d_5 w_2 + d_6 w_3 + d_7 w_4 + \left( \frac{1}{U^*} \right)^2 M(\alpha) = g(\tau) \]

where

\[
c_0 = 1 + \frac{1}{\mu}, \quad c_1 = x - \frac{a_h}{\mu}, \quad c_2 = 2 \xi \frac{\sigma}{U^*} + \frac{2}{\mu} (1 - \psi_1 - \psi_2), \quad c_3 = \frac{1 + 2(\frac{1}{2} - a_h)(1 - \psi_1 - \psi_2)}{\mu},
\]

\[
c_4 = \frac{2}{\mu} (\psi \varepsilon_1 + \psi \varepsilon_2), \quad c_5 = \frac{2}{\mu} \left[ (1 - \psi_1 - \psi_2) + (\frac{1}{2} - a_h)(\psi \varepsilon_1 + \psi \varepsilon_2) \right], \quad c_6 = \frac{2}{\mu} \psi \varepsilon_1 [1 - (\frac{1}{2} - a_h) \varepsilon_1],
\]

\[
c_7 = \frac{2}{\mu} \psi \varepsilon_2 [1 - (\frac{1}{2} - a_h) \varepsilon_2], \quad c_8 = -\frac{2}{\mu} \psi \varepsilon_1^2, \quad c_9 = -\frac{2}{\mu} \psi \varepsilon_2^2,
\]

\[
d_0 = \frac{x_0}{r_a^2 - \frac{a_h}{\mu} r_a^2}, \quad d_1 = 1 + \frac{1 + 8 a_h^2}{8 \mu r_a^2}, \quad d_2 = \frac{2 \xi}{U^*} + \frac{1 - 2 a_h}{2 \mu r_a^2} - \frac{(1 + 2 a_h)(1 - 2 a_h)(1 - \psi_1 - \psi_2)}{2 \mu r_a^2}, \quad d_3 = \frac{(1 + 2 a_h)(1 - \psi_1 - \psi_2)}{\mu r_a^2},
\]

\[
d_4 = \frac{(1 + 2 a_h)(1 - \psi_1 - \psi_2)}{\mu r_a^2}, \quad d_5 = \frac{(1 + 2 a_h)(1 - \psi_1 - \psi_2)}{\mu r_a^2}, \quad d_6 = \frac{(1 + 2 a_h)(\psi \varepsilon_1^2)}{\mu r_a^2},
\]

\[
d_7 = \frac{(1 + 2 a_h)(\psi \varepsilon_2^2)}{\mu r_a^2}, \quad d_8 = \frac{(1 + 2 a_h)(\psi \varepsilon_1 \varepsilon_2)}{\mu r_a^2}, \quad d_9 = \frac{(1 + 2 a_h)(\psi \varepsilon_1^2 \varepsilon_2)}{\mu r_a^2}.
\]

**Appendix B**

Program used for nonlinear aeroelastic calculations, author E. Baldassin, Imperial College London.

Input is given in the file `force.inp`. The entries are given in the following form:

| \( \alpha(0) \) (degrees) |
| \( \alpha'(0) \) (degrees) |
| \( \xi(0) \) |
| \( \xi''(0) \) |
| \( \mu \) |
| \( x_0 \) |
| \( a_h \) |
| \( r_a \) |
| \( \sigma \) |
| \( \xi_0 \) |
| \( \xi_1 \) |
| \( U^*/U_* \) |
| \( \beta_1 \) |
| \( \beta_3 \) |
| \( \beta_5 \) (cubic term for plunge elasticity, usually zero) |
| \( d\tau \) |
| \( \tau_{\text{fin}} \) |
| \( \text{iprint} \) |
| \( d(U^*/U_* \text{fin}) \) |
| \( (U^*/U_* \text{fin}) \) |
where $\tau_{\text{fin}}$ is the integration (nondimensional) time, $\Delta t$ is the time step, $\text{iprint}$ prints the result every $i$ iterations, $d(\frac{U^*/U_L^*}{})$ is the increment in $U^*/U_L^*$ in bifurcation diagrams, and $(U^*/U_L^*)_{\text{fin}}$ is the final value of $U^*/U_L^*$ for the diagram. Below is the code for the program:

```
PROGRAM omega

c Forced oscillation: simulation example in aeroelasticity for a
2 dof, fig.10 pag.241 and fig.12 pag.243

c Dichiarazione variabili

IMPLICIT none

INTEGER*2 i, iconta, iprint, j, n
PARAMETER(n=65)
REAL*8 x(8), xk0(8), xk1(8), xk2(8), xk3(8), xmargin
REAL*8 x1max, x1min, x1amp, x3max, x3min, x3amp, x1old, x2old
REAL*8 w, omgforce(n), xzero(8), PI, dUx, Uxfin, xmaxprev

REAL*8 mu, xa, ah, ra, omg, dampz, dampa, Ux, beta, beta2
COMMON/airfpar/mu, xa, ah, ra, omg, dampz, dampa, Ux, beta, beta2

REAL*8 psi1, psi2, eps1, eps2, beta1
COMMON/costant/ psi1, psi2, eps1, eps2, beta1
REAL*8 rt0, g0, FF
COMMON/forze0/ rt0, g0, FF

REAL*8 time, tfin, dt, time0, time1, time2, time4, time3

REAL*8 alfa3, beta30, beta31, beta32, beta40, beta42, beta43
REAL*8 ck0, ck2, ck3, timeset, UxL, a, xdiff
REAL*8 ft0, g0, FF
COMMON/coeff/ c, d

PI = 3.14159265359

c inizializzazione

FF = .03d0

omgforce(1) = .4d0

DO 5 i=2, n
   omgforce(i) = omgforce(i-1) + .015d0
5 CONTINUE

DO 10 i=1, 8
   x(i) = 0.d0
   xzero(i) = 0.d0
10 continue

time = 0.d0
iconta = 0

psi1 = 0.165d0
```
psi2=0.335d0
eps1=0.0455d0
eps2=0.3d0

lettura dati: condizioni iniziali e parametri del profilo

OPEN(10, file='force.inp', status='old')

condizioni iniziali

do 20 i=1,4
READ(10,*) x(i)
xzero(i) = x(i)
20 continue

do i=1,2
x(i) = x(i)*PI/180.d0
enddo

parametri del profilo

READ(10,*) mu, xa, ah, ra, omg, dampz, dampa, Ux
READ(10,*) betaa1, betaa, betaz

parametri del procedimento numerico

READ(10,*) dt, timeset, iprint
READ(10,*) dUx, Uxfin, UxL

Ux = UxL*Ux
dUx = UxL*dUx
Uxfin = UxL*Uxfin

CLOSE(10, STATUS='KEEP')

tfin = timeset

calcolo della forzante dipendente dalle condizioni iniziali

ft0= 2.d0*((.5d0-ah)*x(1)+x(3))/mu
g0= -(1.d0+2.d0*ah)/(2.d0*ra**2)

OPEN(30, file='timeF', status='unknown')
WRITE(30,*) time
OPEN(31, file='alfaF', status='unknown')
WRITE(31,*) time, (x(1)*180.d0/PI)
OPEN(32, file='alfadotF', status='unknown')
WRITE(32,*) time, (x(2)*180.d0/PI)
OPEN(33, file='zetaF', status='unknown')
WRITE(33,*) time, (x(3)*180.d0/PI)
OPEN(34, file='zetadotF', status='unknown')
WRITE(34,*) time, (x(4)*180.d0/PI)
OPEN(35, file='bifurcationF', status='unknown')
OPEN(36, file='fig10omega.out', status='unknown')
OPEN(37, file='fig10x3.out', status='unknown')
OPEN(38, file='outextra', status='unknown')
OPEN(39, file='magnitudelist', status='unknown')

AN INVESTIGATION OF NONLINEAR AEROELASTICITY IN AIRCRAFT WINGS 43
CALCULO DEI COEFFICIENTI DELLE EQUAZIONI

CALL subcoeff(c,d)

COEFICIENTI PER RK5
alfa3 = 12.d0 / 13.d0
beta30 = 1932.d0 / 2197.d0
beta31 = -7200.d0 / 2197.d0
beta32 = 7296.d0 / 2197.d0
beta40 = 439.d0 / 216.d0
beta42 = 3680.d0 / 513.d0
beta43 = -845.d0 / 4104.d0
ck0 = 25.d0 / 216.d0
ck2 = 1408.d0 / 2565.d0
ck3 = 2197.d0 / 4104.d0

cicle on driving frequency

DO 100 j = 1, n
  time = 0.d0
  xmaxprev = 0.d0
  xmargin = 0.d0
  x1old = 0.d0
  x2old = 0.d0

Runge Kutta integration: RK-5

iconta = iconta + 1

time0 = time
  time1 = time + .25d0 * dt
  time2 = time + .375d0 * dt
  time3 = time + alfa3 * dt
  time4 = time + dt

  time = time4

RK-5 calcola la funzione su 5 punti all'interno dell'intervallo dt

primo punto

CALL subRK5(time0, x, fk0, w)

DO 30 i = 1, 8
  xk0(i) = x(i) + dt * fk0(i) / 4.d0
30 CONTINUE

secondo punto

CALL subRK5(time1, xk0, fk1, w)

DO 40 i = 1, 8
  xk1(i) = x(i) + dt * (.09375d0 * fk0(i) + .28125d0 * fk1(i))
40 CONTINUE

terzo punto
CALL subRK5(time2, xk1, fk2,w)

DO 50 i=1,8
  xk2(i) = x(i) + dt * (beta30*fk0(i) + beta31*fk1(i) + beta32*fk2(i))
50  CONTINUE

c quarto punto

CALL subRK5(time3, xk2, fk3,w)

DO 60 i=1,8
  xk3(i) = x(i) + dt * (beta40*fk0(i) - 8.d0*fk1(i) + beta42*fk2(i) + beta43*fk3(i))
60  CONTINUE

c quinto punto

CALL subRK5(time4, xk3, fk4,w)

c funzione al tempo t+dt

DO 70 i=1,8
  x(i) = x(i) + dt * (ck0*fk0(i) + ck2*fk2(i) + ck3*fk3(i) - 0.2d0*fk4(i))
70  CONTINUE

IF(time.GT.(tfin/2.d0)) THEN
  IF((x(2)*x2old).LT.(0.d0)) THEN
    x1max = x1old + ((x(1)-x1old)*x2old/(x(2)-x2old))
    WRITE(35,*) (Ux/UxL), (x1max*180.d0/PI)
  ENDIF
  xdiff = ABS(x1max - xmaxprev)
  xmaxprev = x1max
  IF(xdiff.GT.xmargin) THEN
    xmargin = xdiff
 ENDIF
ENDIF

IF(mod(iconta,iPrint).EQ.0) THEN
  WRITE(30,*) time,x(1)
  WRITE(31,*) time, (x(1)*180.d0/PI)
  WRITE(32,*) (x(1)*180.0/PI), (x(2)*180.d0/PI)
  WRITE(33,*) time, (x(3)*180.d0/PI)
  WRITE(34,*) time, (x(4)*180.d0/PI)
ENDIF

c ciclo sul tempo

x1old = x(1)
x2old = x(2)
IF(time.LT.tfin) GO TO 25

WRITE(38,*) (Ux/UxL), x(1), x(2), x(3), x(4), x(5), x(6), x(7), x(8)
WRITE(39,*) (Ux/UxL), xmargin

Ux = Ux + dUx

a = Ux/UxL

IF((a.GT.(0.14d0)).and.(a.LT.(0.18d0)))THEN
  tfin = timeset*6.d0
ELSEIF((a.GT.(0.27d0)).and.(a.LT.(0.31d0))) THEN
  tfin = timeset*8.d0
ELSEIF((a.GT.(0.44d0)).and.(a.LT.(0.52d0))) THEN
  tfin = timeset*12.d0
ELSE
  tfin = timeset
ENDIF

DO i=1,8
  x(i) = xzero(i)
ENDDO
PRINT *, a, tfin

IF(Ux.LT.(Uxfin)) GO TO 24

DO i=1,8
  x1amp= (x1max-x1min)/2
  x3amp= (x3max-x3min)/2
  WRITE(35,*) x1amp
  WRITE(36,*) w
  WRITE(37,*) x3amp
ENDDO

100 continue

DO i=1,8
  WRITE(38,*) xzero(i)
ENDDO
CLOSE(30)
CLOSE(31)
CLOSE(32)
CLOSE(33)
CLOSE(34)
CLOSE(35)
CLOSE(36)
CLOSE(37)
CLOSE(38)
CLOSE(39)
STOP
SUBROUTINE subcoeff(c,d)
  c  calcolo dei coefficienti per le equazioni del moto per qualunque
  c  tipo di non linearita'

IMPLICIT none

REAL*8 mu, xa, ah, ra, omg, dampz, dampa, Ux, betaa, betaz
COMMON/airfpar/mu, xa, ah, ra, omg, dampz, dampa, Ux, betaa, betaz

REAL*8 c(10), d(10)

REAL*8 psi1, psi2, eps1, eps2, betaa1
COMMON/costant/ psi1, psi2, eps1, eps2, betaa1

REAL*8 a, u, p1p2, pe12, pe1, pe2, r, ap, am

c  termini ricorrenti
  a= .5d0-ah
  u= 2.d0/mu
  p1p2= 1.d0-psi1-psi2
  pe1= psi1*eps1
  pe2= psi2*eps2
  pe12= pe1+pe2
  r= 1.d0/(mu*ra**2)
  ap= 1.d0+2.d0*ah
  am= 1.d0-2.d0*ah

c  calcolo dei coefficienti ci
  c(1)= 1.d0+1.d0/mu
  c(2)= xa-ah/mu
  c(3)= (2.d0*dampz*omg/Ux)+u*p1p2
  c(4)= (1.d0+2.d0*a*p1p2)/mu
  c(5)= u*pe12
  c(6)= u*(p1p2+a*pe12)
  c(7)= u*pe1*(1.d0-a*eps1)
  c(8)= u*pe2*(1.d0-a*eps2)
  c(9)= -u*pe1*eps1
  c(10)= -u*pe2*eps2

c  calcolo coefficienti di
  d(1)= -ah*r*xa/ra**2
  d(2)= 1.d0+r*(1.d0+8.d0*ah**2)/8.d0
  d(3)= (2.d0*dampa/Ux)+am*r*5.d0-5.d0*r*ap*am*p1p2
  d(4)= -r*ap*p1p2-5.d0*r*ap*am*pe12
  d(5)= -r*ap*p1p2
  d(6)= -ap*pe12*r
  d(7)= -r*ap*pe1*(1.d0-a*eps1)
  d(8)= -r*ap*pe2*(1.d0-a*eps2)
  d(9)= r*ap*pe1*eps1
  d(10)= r*ap*pe2*eps2

RETURN
END

SUBROUTINE subRK5(time,x,fk,w)
c integrazione numerica di Runge Kutta 5 (non linearita' di tipo cubico)

IMPLICIT none
REAL*8 c(10), d(10)
COMMON/coeff/ c, d

REAL*8 Gx3, Mx1, P, H, w, FF
REAL*8 omg, Ux, betaa, betaz
REAL*8 mu, xa, ah, ra, dampz, dampa
COMMON/airfpar/mu, xa, ah, ra, omg, dampz, dampa, Ux, betaa, betaz

REAL*8 ft, ft0, g, g0, Pt, Qt
COMMON/forze0/ ft0, g0, FF
REAL*8 psi1, psi2, eps1, eps2, betaa1
COMMON/costant/ psi1, psi2, eps1, eps2, betaa1

REAL*8 x(8), time, fk(8)

Tipo di non linearita': cubica

Gx3 = x(3)+betaz*x(3)**3
Mx1 = betaa1*x(1)+betaa*x(1)**3

c calcolo delle forzanti

Pt = 0.d0
Qt = 0.d0

ft = ft0*(psi1*eps1*EXP(-eps1*time)+ psi2*eps2*EXP(-eps2*time))+ Pt
gt = g0*ft + Qt - Pt*g0

c coefficienti P e H

P = c(3)*x(4)+c(4)*x(2)+c(5)*x(3) +c(6)*x(1)+c(7)*x(5)+c(8)*x(6)+
   c(9)*x(7)+c(10)*x(8)+Gx3*(omg/Ux)**2 - ft

H = d(3)*x(2)+d(4)*x(1)+d(5)*x(4)+d(6)*x(3)+d(7)*x(5)+d(8)*x(6)+
   d(9)*x(7)+d(10)*x(8)+Mx1*(1/Ux)**2 - gt

c equazioni del sistema

fk(1)= x(2)
fk(2) = (c(1)*H+d(1)*P)/(d(1)*c(2)-c(1)*d(2))
fk(3)= x(4)
fk(4) = (-c(2)*H+d(2)*P)/(d(1)*c(2)-c(1)*d(2))
fk(5)= x(1)-eps1*x(5)
fk(6)= x(1)-eps2*x(6)
fk(7)= x(3)-eps1*x(7)
fk(8)= x(3)-eps2*x(8)

RETURN

END
Praca dotyczy badania właściwości nieliniowych modeli sprężystych opisujących ruch skrzydeł płatowców. Modele ruchu skrzydeł stworzone zostały w oparciu o równania różniczkowe, do rozwiązania których wykorzystano metodę Runge-Kutta-Fehlberg’a. Walidacja wykonana w oparciu o przykłady zawarte w literaturze wykazała poprawność przyjętych modeli.

Ostatni etap pracy dotyczy badania wpływu zmiany parametrów modelu sprężyny trzeciego stopnia na amplitudę ruchu skrzydła po perturbacji. Badania wykazały korelację między twardszą sprężyną i mniejszą amplitudą cyklu granicznego drgającego skrzydła.